# Analyses of four-bar linkages through Multibody Dynamics Approach

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**ABSTRACT**

This paper presents a recursive algorithm for kinematic and dynamic analyses of a closed-loop system, namely, a four-bar linkage. A minimal set of constrained dynamic equations of motion are obtained using the decoupled natural orthogonal complement (DeNOC) matrices from the uncoupled Newton-Euler equations. These constrained equations contain the Lagrange multipliers representing the loop-closure reactions. Lagrange multipliers are solved first. Reactions at other joints are then determined recursively using the dynamic equilibrium equations of each link of the linkage at hand.

## 1. Introduction

Four-bar linkages are widely used in mechanical devices owing to their simplicity of structure, ease of manufacturing, and low cost. Kinematic synthesis, dynamic design, and optimization are the essential components in their design process [1]. The process is highly iterative to reach at satisfactory design. Traditionally, kinematic and dynamic analyses have been done for few critical configurations using any of graphical or analytical techniques. Analytical techniques depend on the formulation of dynamical equations of motion and their solution methodology [2]. In this paper, a recursive formulation is proposed for the kinematic and dynamic analyses of four-bar linkages. It is virtually opened by cutting a loop-closure-joint. Then, the dynamic equations are derived in Cartesian coordinates [3]. Finally, using the decoupled natural orthogonal complement (DeNOC) matrices associated with the velocity constraints of the connected links, the dimension of dynamic equations is reduced [4-6]. Note here that the relative velocities and accelerations of the unactuated joints are determined recursively form the actuated one [7]. Since the reaction forces at the cut-joint are nothing but the Lagrange multipliers [8], they are calculated first from the dynamic equations. Next, it is shown that the reactions at the other joints can be obtained recursively by writing the force equilibrium equations for each link [9].

## 2. Definitions of Twist and Wrench

The 6-dimensional vectors of twist and wrench of the ith body shown in Fig. 1 is defined as

$$\mathbf{t}_i \equiv \begin{bmatrix} \boldsymbol{\omega}_i \\ \mathbf{v}_i \end{bmatrix} \text{ and } \mathbf{w}_i \equiv \begin{bmatrix} \mathbf{n}_i \\ \mathbf{f}_i \end{bmatrix} \tag{1}$$

where $\boldsymbol{\omega}_i$ and $\mathbf{v}_i$ are the 3-dimensional vectors of angular and linear velocity of the origin point, $O_i$, of the ith body, respectively, whereas, $\mathbf{n}_i$ and $\mathbf{f}_i$ are the 3-dimensional vectors of the moment about $O_i$ and the force at $O_i$, respectively. Now, the twist, $\mathbf{t}_i$, can be obtained from its previous one, i.e., the *(i-1)*st one [5], as

$$\mathbf{t}_i = \mathbf{A}_{i,i-1}\mathbf{t}_{i-1} + \mathbf{p}_i\dot{\theta}_i \tag{2}$$

where the 6×6 twist propagation matrix, $\mathbf{A}_{i,i-1}$ , and the 6-dimensional joint-motion-propagation vector are given by

$$\mathbf{A}_{i,i-1} \equiv \begin{bmatrix} \mathbf{1} & \mathbf{O} \\ \mathbf{a}_{i,i-1} \times \mathbf{1} & \mathbf{1} \end{bmatrix}; \text{ and } \mathbf{p}_i \equiv \begin{bmatrix} \mathbf{e}_i \\ \mathbf{0} \end{bmatrix} \text{ for revolute}$$

$$\mathbf{p}_i \equiv \begin{bmatrix} \mathbf{0} \\ \mathbf{e}_i \end{bmatrix} \text{ for prismatic} \qquad (3)$$

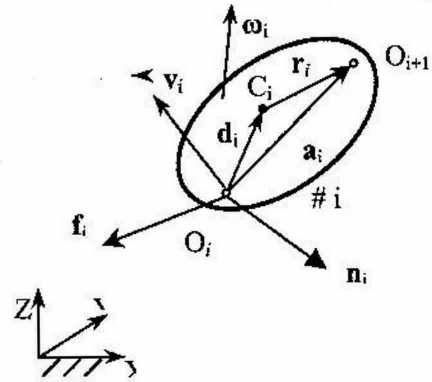


Fig. 1 Definition of various vectors

In eq. (3), $\mathbf{a}_{i,i-1}$ is the vector from $O_i$ to $O_{i-1}$. For a link with two revolute joints at its two ends, $\mathbf{a}_{i,i-1} \equiv \mathbf{a}_{i-1} = \mathbf{d}_{i-1} + \mathbf{r}_{i-1}$. Moreover, **O** and **1** are the 3x3 matrix of zeros and the 3´3 identity matrix, respectively, whereas, 0 the 3-dimensional vector of zeros, and $e_i$ the unit vector parallel to the ith joint axis.

## 3 Recursive Relations for Unactuated Joint Rates and Twists

Links of the four-bar linkage, shown in Fig. 2, are numbered as #1, ..., #4, where #4 is the fixed link. The revolute joint between link #4 and #1 is numbered as 1, and similarly joints 2, 3 and 4 are represented in Fig. 2. Positions of the mass centers of each link are indicated by vectors $\mathbf{d}_i$ and $\mathbf{r}_i$, for i=1, 2, 3, from the respective joints. The link lengths are defined by the magnitude of the vector, $\mathbf{a}_i$, for i=1, 2, 3. Three joint coordinates, $\theta_i$, and the link lengths, $a_i$, for i=1, 2, 3, are required to define the system's configuration unambiguously. Fourth joint coordinate, $\theta_4$, is redundant, as $\theta_4 = 4\pi - (\theta_1 + \theta_2 + \theta_3)$. The four-bar linkage has one-degree-of-freedom (dof). Hence, it can be controlled by a single actuator only. Let us assume that the actuator is located at $O_1$, Fig. 2. Other joints, 2, 3, and 4, are the unactuated joints whose joint-rates are obtained next recursively using the methodology proposed in [7]. We do not give complete derivation of the joint velocities and accelerations to save space. However, one can obtain the following expressions using eq. (2):

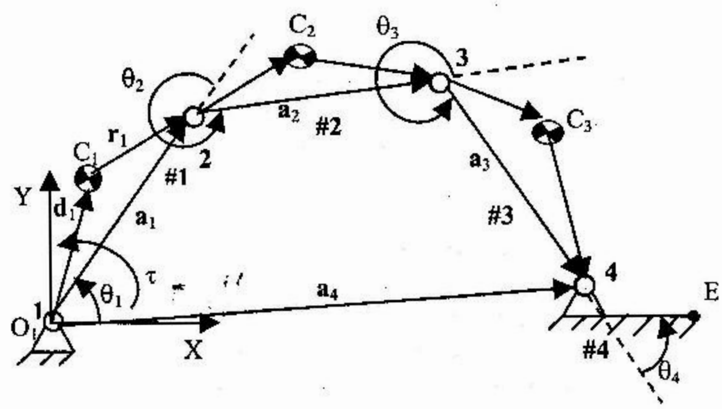


Fig. 2 A four-bar linkage

$$\dot{\theta}_i = \frac{\tilde{\mathbf{p}}_i^T}{\delta_i}(\mathbf{t}_e - \mathbf{A}_{e,i-1}\mathbf{t}_{i-1}), \text{ for i=2, 3, 4} \qquad (4)$$

where

$$\tilde{\mathbf{p}}_i \equiv \mathbf{\Phi}_{i+1}\mathbf{A}_{ei}\,\mathbf{p}_i;\ \delta_i \equiv \tilde{\mathbf{p}}_i^T\tilde{\mathbf{p}}_i;\ \mathbf{\Phi}_i \equiv \mathbf{\Phi}_{i+1} - \frac{\tilde{\mathbf{p}}_i\tilde{\mathbf{p}}_i^T}{\delta_i};\text{ and } \mathbf{\Phi}_5 = \mathbf{1} \qquad (5)$$

Like a serial chain arm's end-effector, point E on link #4 is assumed as the end-effector whose velocities vanish, i.e., the twist, $\mathbf{t}_e = \mathbf{t}_4 = \mathbf{t}_0$. The joint acceleration of each link is similarly determined by differentiating the twist expressions with respect to time and rearranging as

$$\ddot{\theta}_4 = \frac{\tilde{\mathbf{p}}_4^T}{\delta_4}[\dot{\mathbf{t}}_e - \dot{\mathbf{A}}_{e4}\mathbf{t}_4 - \mathbf{A}_{e4}(\dot{\mathbf{A}}_{43}\mathbf{t}_3 + \mathbf{A}_{43}\dot{\mathbf{t}}_3 + \dot{\mathbf{p}}_4\dot{\theta}_4)] \qquad (6)$$

$$\ddot{\theta}_3 = \frac{\tilde{\mathbf{p}}_3^T}{\delta_3}[\Phi_4(\dot{\mathbf{t}}_e - \alpha_1) - \alpha_2] \text{ and } \ddot{\theta}_2 = \frac{\tilde{\mathbf{p}}_2^T}{\delta_2}[\Phi_3(\dot{\mathbf{t}}_e - \alpha_3) - \alpha_2] \qquad (7)$$

where $\alpha_3 \equiv \mathbf{A}_{e3}[\dot{\mathbf{A}}_{32}\mathbf{t}_2 + \mathbf{A}_{32}(\dot{\mathbf{A}}_{21}\mathbf{t}_1 + \mathbf{A}_{21}\dot{\mathbf{t}}_1 + \dot{\mathbf{p}}_2\dot{\theta}_2) + \dot{\mathbf{p}}_3\dot{\theta}_3 + \dot{\mathbf{A}}_{e3}\mathbf{t}_3]$. Note that $\alpha_1 = \alpha_3 + \mathbf{A}_{e2}\mathbf{p}_2\ddot{\theta}_2$.

## 5. Dynamic Analysis

In this section, a novel dynamic formulation for four-bar linkages is presented for the determination of constraint forces and moments at the joints. First, the four-bar linkage is opened by virtually cutting a joint. One can cut any of the four joints of the linkage. Then uncoupled Newton-Euler equations of motion for the resulting open system are written from the free-body diagrams in the Cartesian coordinates. Sequentially, decoupled natural orthogonal complement (DeNOC) [6] matrix associated to the open system is used to reduce the dimension of the equations of motion leading to the determinate set of equations in terms of the constraint forces of the cut-joint plus driving torque. Knowing the constraint forces at the cut-joint, constraint forces at other joints are then determined recursively from the end body to the first body fixed to the base. The proposed methodology is efficient as majority of the constraint forces and moments are evaluated recursive. Since the four-bar linkageis a closed-loop system, the joint co-ordinates $\theta_1, \theta_2$, and $\theta_3$ are not independent due to the loop closure constraint. First, they are made independent by cutting the joint 4.

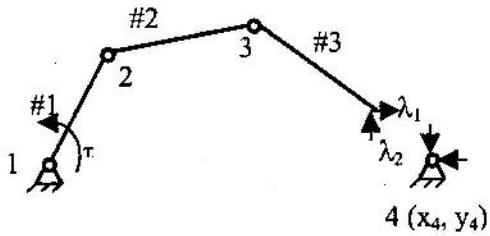



Fig. 3 Open system of four bar linkage

Now, the 18-dimensional generalized twist, $\mathbf{t} \equiv [\mathbf{t}_1^T,\ \mathbf{t}_2^T,\ \mathbf{t}_3^T]^T$, of the open system is obtained from eq. (2) as

$$\mathbf{t} = \mathbf{N}\dot{\theta}, \text{ where } \mathbf{N} \equiv \mathbf{N}_l\mathbf{N}_d \tag{8}$$

The 18×18 lower block triangular matrix, $N_l$, and the 18×3 block diagonal matrix, $N_d$, are as follows:

$$\mathbf{N}_l \equiv \begin{bmatrix} 1 & O & O \\ \mathbf{A}_{21} & 1 & O \\ \mathbf{A}_{31} & \mathbf{A}_{32} & 1 \end{bmatrix};\ \mathbf{N}_d \equiv \begin{bmatrix} \mathbf{p}_1 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{p}_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{p}_3 \end{bmatrix};\ \text{and } \mathbf{N} = \begin{bmatrix} \mathbf{p}_1 & \mathbf{0} & \mathbf{0} \\ \mathbf{A}_{21}\mathbf{p}_1 & \mathbf{p}_2 & \mathbf{0} \\ \mathbf{A}_{31}\mathbf{p}_1 & \mathbf{A}_{32}\mathbf{p}_2 & \mathbf{p}_3 \end{bmatrix} \tag{9}$$

The matrices $N_l$ and $N_d$ are the DeNOC matrices, whereas N is called the NOC matrix. The Newton-Euler (NE) equations of motion for the ith rigid body with respect to the origin, $O_i$, are then expressed as [7]

$$\mathbf{M}_i\dot{\mathbf{t}}_i + \mathbf{W}_i\mathbf{M}_i\mathbf{E}_i\mathbf{t}_i = \mathbf{w}_i \tag{10}$$

where the 6×6 matrices of the extended mass, $M_i$, and of the extended angular velocity, $W_i$, and the 6×6 coupling matrix, $E_i$ are defined as

$$\mathbf{M}_i \equiv \begin{bmatrix} \mathbf{I}_i & m_i\mathbf{D}_i \\ -m_i\mathbf{D}_i & m_i\mathbf{1} \end{bmatrix};\ \mathbf{W}_i \equiv \begin{bmatrix} \Omega_i & O \\ O & \Omega_i \end{bmatrix};\ \text{and } \mathbf{E}_i \equiv \begin{bmatrix} \mathbf{1} & O \\ O & O \end{bmatrix} \tag{11}$$

in which $\mathbf{D}_i$ and $\Omega_i$ are the 3 x 3 cross-product tensors associated with the 3-dimensional vectors, $\mathbf{d}_i$ and $\omega_i$, respectively, which are defined as, $\mathbf{D}_i\mathbf{x} \equiv \mathbf{d}_i \times \mathbf{x}$ and $\Omega_i\mathbf{x} \equiv \omega_i \times \mathbf{x}$, for any 3-dimensional Cartesian vector, $\mathbf{x}$. Vector $\mathbf{n}_i$ is the resultant moment about, $O_i$, and $\mathbf{f}_i$ is the resultant force at $O_i$, as shown in Fig. 1.

In order to make the system equivalent to the closed-loop system the cut-joint forces, known as the Lagrange multipliers, are introduced at the cut-joint. Writing the NE equation, eq.(10), for i=1, 2, and 3, gives 18 uncoupled scalar equations of motion for the four-bar linkage as

$$\mathbf{M}\dot{\mathbf{t}} + \mathbf{WMEt} = \mathbf{w} \tag{12}$$

where the 18 x18 matrices, **M, W,** and **E** are the generalized mass, angular velocity, and coupling matrices, respectively, i.e.,

$$\mathbf{M} \equiv \mathrm{diag}[\mathbf{M}_1, \mathbf{M}_2, \mathbf{M}_3] \ \mathbf{W} \equiv \mathrm{diag}[\mathbf{W}_1, \mathbf{W}_2, \mathbf{W}_3] \ \text{; and } \mathbf{E} \equiv \mathrm{diag}[\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3] \tag{13}$$

Moreover, the 18-dimensional generalized wrench vector, $\mathbf{w}$, is defined as $\mathbf{w} \equiv [\mathbf{w}_1^T, \ \mathbf{w}2_1^T, \ \mathbf{w}_3^T]^T$. The expressions in the left hand side of eq. (12) denote the effective inertial forces and moments, and those on the right hand side are the external forces and moments. The wrench, $\mathbf{w}$, can be split into three part, namely, the wrench due to externally applied force $\mathbf{w}^e$, wrench of constraint forces due to joints, $\mathbf{w}^c$, and the wrench of the Lagrange multipliers due to the cut-joint at joint 4, $\mathbf{w}^\lambda$. Therefore, $\mathbf{w} \equiv \mathbf{w}^e + \mathbf{w}^\lambda + \mathbf{w}^c$. Premultiplying eq. (12) by $\mathbf{N}^T$ leads to

$$\mathbf{N}^T(\mathbf{M}\dot{\mathbf{t}} + \mathbf{WMEt}) = \mathbf{N}^T(\mathbf{w}^e + \mathbf{w}^\lambda + \mathbf{w}^c) \tag{14}$$

Equation (14) is constrained NE equation of motion. Since for independent $\dot{\theta}$, $\mathbf{N}^T\mathbf{w}^c = 0$, because the constraint forces and moments do not perform any work, i.e., $\mathbf{t}^T\mathbf{w}^c = \dot{\theta}^T\mathbf{N}^T\mathbf{w}^c = 0$ eq.(17) is rewritten as

$$\mathbf{N}^T(\mathbf{M}\dot{\mathbf{t}} + \mathbf{WMEt}) = \mathbf{N}^T\mathbf{w}^e + \mathbf{N}^T\mathbf{w}^\lambda \tag{15}$$

For the planar four-bar linkage having all revolute joints, eq. (15) is converted into three scalar equations with three unknowns, namely, two Lagrange multipliers at the cut-joint and one driving torque applied at joint 1. Hence, the constrained NE equation of motion, eq. (15) are solvable. Note here that the wrench associated with the Lagrange multipliers, $w^\lambda$, for the planar open four-bar linkage defined as

$$\mathbf{w}^\lambda \equiv \begin{bmatrix} \mathbf{w}_1^{\lambda^T} & \mathbf{w}_2^{\lambda^T} & \mathbf{w}_3^{\lambda^T} \end{bmatrix}^T \tag{16}$$

where the joints associated with links #1 and #2 are not cut. Hence, $\mathbf{w}_1^\lambda = \mathbf{w}_2^\lambda = 0$. For the other link, #3, joints 4 (between #3 and #4) is cut. Hence, $\mathbf{w}_3^\lambda$ with respect to its origin point, $O_3$, is obtained as

$$\mathbf{w}_3^\lambda \equiv \mathbf{A}'_{34}\mathbf{w}_4^\lambda, \text{ where } \mathbf{w}_4^\lambda \equiv [0,0,0,\lambda_1,\lambda_2,0]^T \text{ and } \mathbf{A}'_{34} \equiv \begin{bmatrix} \mathbf{1} & \mathbf{a}_{43} \times \mathbf{1} \\ \mathrm{O} & \mathbf{1} \end{bmatrix} \tag{17}$$

In which the 6x6 matrix, $\mathbf{A}'_{34}$, is the wrench propagation matrix. The generalized wrench associated with the Lagrange multipliers, $w^\tau$, eq. (16), is now expressed as

$$\mathbf{w}^\lambda = \mathbf{A}'\mathbf{E}'\lambda \tag{18}$$

where the 18 x 6 matrix, $\mathbf{A}'$, the 6 x 2 matrix, $\mathbf{E}'$, and the 2-dimensoinal vector $\lambda$, are as follows:

$$\mathbf{A}' \equiv \begin{bmatrix} \mathrm{O} \\ \mathrm{O} \\ \mathbf{A}'_{34} \end{bmatrix}; \quad \mathbf{E}' \equiv \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}^T; \text{ and } \lambda \equiv \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} \tag{19}$$

**O** being the 6 x 6 zero matrix, whereas $\lambda_1$ and $\lambda_2$ are the Lagrange multipliers representing the reaction forces at joint 4 along the X and Y-axes, respectively, as shown in Fig. 3. The generalized forces and moments due to the Lagrange multipliers, $\tau^\lambda$, is then obtained as

$$\tau^\lambda = \mathbf{N}^T\mathbf{w}^\lambda = \mathbf{J}^T\lambda \text{ , where } \mathbf{J} \equiv \mathbf{E}'^T\mathbf{A}'^T\mathbf{N}_l\mathbf{N}_d \tag{20}$$

Upon substitution of the expressions of the DeNOC matrices from eq. (9) and that of matrix, $\mathbf{A}'$, from eq. (19), the 2 x 3 matrix, J, is given by

$$\mathbf{J} = \mathbf{E}'^T[\mathbf{A}_{41}\mathbf{p}_1 \quad \mathbf{A}_{42}\mathbf{p}_2 \quad \mathbf{A}_{43}\mathbf{p}_3] \tag{21}$$

where the following properties of the twist and wrench propagation matrices [9] are used:

$$\mathbf{A}_{i,j}^T = \mathbf{A}'_{j,i}\text{, and } \mathbf{A}'_{i,j}\mathbf{A}'_{j,k} = \mathbf{A}'_{i,k} \tag{22}$$

Using the associated vector and matrix expressions, i.e,

$$\mathbf{e}_i \equiv \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\text{; and } \mathbf{a}_{i,j} \times 1 \cong \begin{bmatrix} 0 & 0 & a_{i,jy} \\ 0 & 0 & -a_{i,jx} \\ -a_{i,jy} & a_{i,jx} & 0 \end{bmatrix} \tag{23}$$

for i=1,2, and 3, and j=4. The 2 x 3 matrix, **J**, eq. (21), is simplified for the planar four-bar linkage, Fig. 3, as

$$\mathbf{J} = \begin{bmatrix} -a_{14y} & -a_{24y} & -a_{34y} \\ a_{14x} & a_{24x} & a_{34x} \end{bmatrix} \tag{24}$$

The terms, $a_{i,jx}$ and $a_{i,jy}$, are the components of vector $\mathbf{a}_{i,j}$ along X and Y axes, respectively. The matrix, **J**, is nothing but the constraint Jacobian which can be verified by the writing the loop-closure equations [3] associated to the four-bar linkage.

Similarly, the generalized external forces defined as, $\boldsymbol{\tau}^e \equiv \mathbf{N}^T\mathbf{w}^e$, can be obtained systematically, whose, $\mathbf{w}^e$, is given by

$$\mathbf{w}^e \equiv \begin{bmatrix} \mathbf{w}_1^{e^T} & \mathbf{w}_2^{e^T} & \mathbf{w}_3^{e^T} \end{bmatrix}^T \tag{25}$$

and $\mathbf{w}_1^e \equiv [0 \quad 0 \quad \tau \quad 0 \quad 0 \quad 0]^T$; and $\mathbf{w}_2^e = \mathbf{w}_3^e = 0$. Here, it is assumed that all joints are frictionless and the motion is on a horizontal plane, i.e., no gravity effect is present. Using the expressions for the DeNOC matrices, eq. (9), the generalized external forces is obtained as

$$\boldsymbol{\tau}^e = \begin{bmatrix} \mathbf{w}_1^{e^T}\mathbf{p}_1 & 0 & 0 \end{bmatrix}^T = [\tau \quad 0 \quad 0]^T \tag{26}$$

Finally, substituting $\tau^\lambda$ and $\tau^e$, eqs. (20) and (26), respectively, into eq. (15), one can obtain the unknown Lagrange multipliers and the driving torque by solving

$$\begin{bmatrix} 1 & -a_{14y} & a_{14x} \\ 0 & -a_{24y} & a_{24x} \\ 0 & -a_{34y} & a_{34x} \end{bmatrix} \begin{bmatrix} \tau \\ \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} n_1^* + n_2^* - \mathbf{a}_{12}^T\mathbf{E}\mathbf{f}_2^* + n_3^* - (\mathbf{a}_{12}^T + \mathbf{a}_{23}^T)\mathbf{E}\mathbf{f}_3^* \\ n_2^* - \mathbf{a}_{23}^T\mathbf{E}\mathbf{f}_3^* \\ n_3^* \end{bmatrix} \tag{27}$$

where $n_i^*$ and $\mathbf{f}_i^*$, for i=1, 2, 3, are the scalar inertia moment and the force vector determined using the left hand side of eq. (10), respectively, whereas E is the 2´2 planar rotation matrix by 90o that takes care of the result of a vector cross-product in planar motion. Matrix E is defined as

$$\mathbf{E} \equiv \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

The constraint forces and moments at the other joints, namely, at joints 1, 2, and 3, can be computed using the recursive formulation proposed in [9]. Hence, all the forces and moments acting on all the links and joints are determined, which are useful for the design of links and joints, force and moment optimization, and any other dynamic applications.

## 6. Numerical Example

The Hoeken's four-bar mechanism used in the carpet scrapping machine that has been designed and developed at IIT Delhi [10] is taken here for the illustration purposes. The mechanism is shown in Fig. 4, whereas its link parameters are given in the Table 1. The input motion provided to link 1 is a constant speed of 45 rpm (4.7124 rad/sec) whereas the coupler point C traces a straight line. The fixed inertial frame, XYZ, is located at joint 1, i.e., O1. Kinematic analysis is performed using the recursive joint rates and accelerations discussed in Sections 3 and 4. Position analysis is resolved using the loop closure equations, given in the text books on mechanisms, say [11]. The results of the kinematic analyses, namely, $,\theta_i$, $\dot{\theta}_i$, and $\ddot{\theta}_i$, for i=1, ..., 4, are given in Fig. 5.

The constraint forces are obtained using the methodology presented in Section 5. The Lagrange multipliers, namely, $\lambda_1$ and $\lambda_2$, and the driving torque, $\tau$, related to the motion constraint are evaluated first using the constrained NE equations of motion, eq. (15), and shown in Fig. 6. Reaction forces (Fig. 7) at other joints, namely, at 1, 2, and 3, are then evaluated using the recursive force and moment balance equations. The above results are also verified using ADAMS 2005 (Automated Dynamic Analysis of Mechanical Systems) software [12]. The results match exactly.

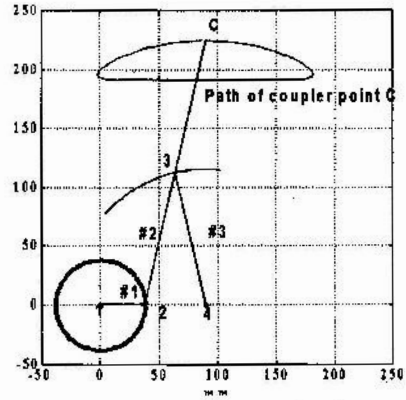
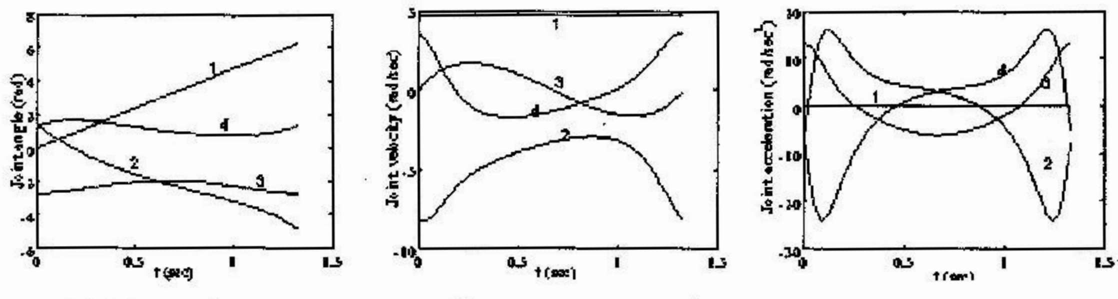



Fig. 4 The Hoeken's mechanism

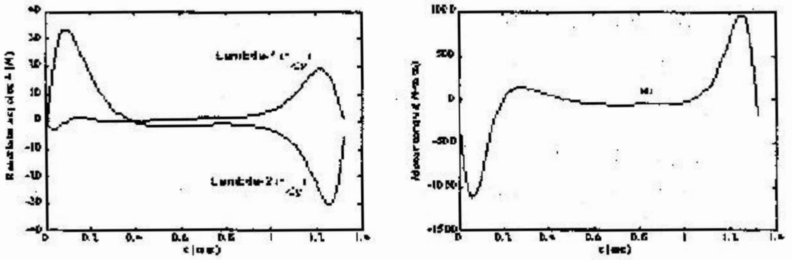
**Table 1 Link parameters**

| *Link No.* | *Length, $a_i$ (mm)* | *Mass, $m_i$ (Kg)* | *Moment of inertia, $I_i$ (Kg-mm$^2$)* | *Initial Position, $\theta_i$ (degree)* |
|---|---|---|---|---|
| 1 | 38.1 | 1.5 | 725.80 | 0 |
| 2 | 115.2 | 5.0 | 88473.60 | 77.12 |
| 3 | 115.2 | 3.0 | 13271.76 | -154.24 |
| 4 | 89.5 | - | - | 0 |



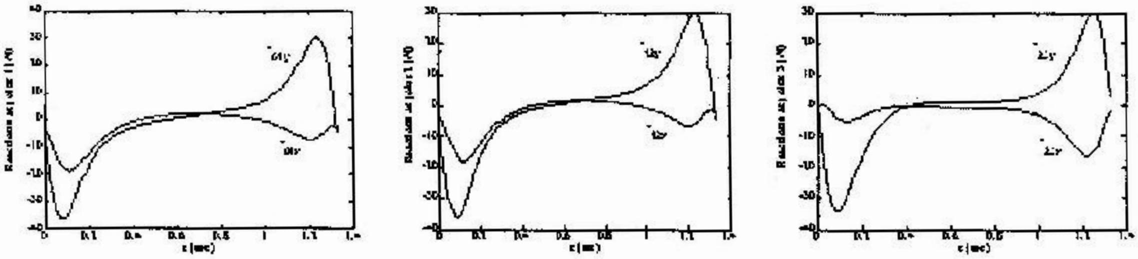

(a) Joint angles, $\theta_i$ (b) Joint velocity, $\dot{\theta}_i$ (c) Joint acceleration, $\ddot{\theta}_i$

Fig. 5 Kinematic analysis of Hoeken's mechanism



(a) Reaction at cut-joint 4 (b) Driving torque

Fig. 6 The Lagrange multipliers and driving torque



(a) Reaction at joint 1 (b) Reaction at joint 2 (c) Reaction at joint 3

Fig. 7 Reaction forces at other joints

## 7. Conclusions

A recursive algorithm for the velocity and acceleration analyses of a closed-loop system, namely, the four-bar linkage is presented. Besides, the dynamics algorithm is separated into two layers. In the first layer, constrained dynamic equations of motion using the Lagrange multipliers that represent the cut-joint reactions and the driving torque are solved simultaneously, followed by the recursive evaluation of the rest of the constraint forces. This way, simultaneous solution all the uncoupled Newton-Euler (NE) equations of motion is avoided. Hence, the efficiency and numerical accuracy of finding the constraint forces are improved.

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