Identification of Crack Model Parameters in a Beam from Modal Parameters

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Abstract

An identification method has been proposed for obtaining crack parameters, mainly flexibility coefficients and as a by-product the equivalent crack depth, in a cracked beam. Euler-Bernoulli beam theory is considered in modeling of the beam. A transverse crack is assumed to remain open and is modeled by five flexibility coefficients in accordance with the linear fracture mechanics approach. The proposed identification method relies on modal parameters (i.e. natural frequencies and mode shapes) and location of the crack for estimation of crack parameters. Numerical simulations have been presented to show the applicability of the method. The finite element method (FEM) is used to find out the transverse natural frequencies and mode shapes of the cracked beam. Identified crack flexibility coefficients have been used to obtain the equivalent crack depth ratio in conjunction with Newton-Raphson method. The identified crack flexibility coefficients and crack depth ratio are in well agreement with assumed ones.

Keywords: Modal parameters, Identification, Crack parameters.

1. Introduction

Crack is a damage that often occurs in structural members and may cause serious failure of the structures. A crack must be detected in the early state. However, it is difficult to recognize a crack by using visual inspection techniques, and it may be detected usually by non-destructive techniques. Knowing the dynamic behavior of a structure with cracks is of significant importance in engineering. There are two types of problems related to this topic: the first may be called "direct problem" and the second called "inverse problem". The "direct problem" is to determine the effect of damages on the structural dynamic characteristics, while the "inverse problem" is to detect, locate and quantify the extent of the damages. In the past three decades, both the direct and inverse problems have attracted many researchers and many relevant literatures have been published.

There are lots of literatures that deal with crack modeling, natural frequencies and mode shape analysis for transverse and longitudinal vibrations of the cracked beam. Wauer [1] presented a review of literatures in the field of dynamics of cracked rotors, including the modeling of the cracked part of structures and determination of different detection procedures to diagnose fracture damages. Gasch [2] provided a survey of the stability behaviour of a rotating shaft with a crack and of the forced vibrations due to imbalances. Dimarogonas [3] reviewed the analytical, numerical and experimental investigations on the detection of a structural flaw based on the changes in dynamic characteristics. Doebling *et al.* [4] provided an overview of methods to detect, locate, and characterize damage in structural and mechanical systems by examining changes in measured vibration response. Salaw [5] reviewed the use of natural frequency as a diagnostic parameter in structural assessment procedures using vibration monitoring. Factors (like, choice of measuring points, effects of ambient conditions on dynamic response and consistency and reliability of the testing procedure, etc.) which could limit successful application

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of vibration monitoring to damage detection and structural assessment were also discussed. Recently, Giridhar *et al.* [6] presented the review of literatures published since 1990 and some classical papers on crack detection and estimation of its severity in shafts. The review was based on three categories namely vibration-based methods, modal testing and non-traditional methods.

Most of detection and diagnostic methods of crack are available in literatures based on *feature extractions* of the free and forced responses, which becomes very complicated and difficult to use in practice. Recently, Tiwari and coworkers [7-9] developed an algorithm for identifying the crack flexibility coefficients and subsequently estimation of the equivalent crack depth based on the force-response information. Based on the physical consideration of the problem reduction schemes were outlined for eliminating the rotational degree of freedoms at crack element nodes, which is otherwise difficult to condense with conventional condensation schemes.

In the present paper the natural frequency and mode shape informations have been used to obtain the crack flexibility coefficients and subsequently size of the crack. It is assumed that the crack location is known. The crack flexibility coefficients are used to obtain the equivalent crack depth by minimizing an error function with the help of Newton-Raphson method. Numerical experiments were conducted to identify the crack parameters. The crack flexibility coefficients and crack size identified by the present algorithm closely agree with the assumed parameters.

2. System modeling

The Euler-Bernoulli beam theory is used for the transverse vibration of the beam. The FEM is used to develop the beam model. The crack is assumed to be an open crack in the present analysis i.e. linear analysis has been performed. Only single crack has been considered in the beam. The free vibration of the beam is considered. The spinning of the beam has not been considered. For a cracked beam element the flexibility can be expressed by a full 6×6 compliance matrix corresponding to a general loading as shown in Fig. 1. For the present analysis axial and torsional effects have not been considered. Corresponding form of the 4×4 flexibility matrix is (A detailed expressions of all flexibility coefficients are given in the [7]) give as

$$[C_c]^{(e)} = \begin{bmatrix} C_{22} & 0 & 0 & 0 \\ & C_{33} & 0 & 0 \\ & & C_{44} & C_{45} \\ \text{sym} & & & C_{55} \end{bmatrix} \quad \text{Where } C_{ij} \text{ is the crack flexibility coefficient.} \tag{1}$$

3. System equations of motion

The Euler-Bernoulli beam is descretized into number of elements, the equation of motion for the beam element without crack for free vibrations becomes

$$[M]^{(e)}\{\ddot{q}(t)\}^{(e)} + [K_{wc}]^{(e)}\{q(t)\}^{(e)} = \{0\} \tag{2}$$

Where $[m]^{(e)}$ is the element mass matrix, $[K_{wc}]^{(e)}$ is the element stiffness matrix and $\{q(t)\}^{(e)}$ is the element vector of nodal degrees of freedom (dofs). The subscripit '*wc*' represent *without crack* and the superscript '*e*' represents *element*. Details of the mass and stiffness matrices are given in [7]. The equation of motion of a cracked beam element can be expected as

$$[M]^{(e)}\{\ddot{q}_c(t)\}^{(e)} + [K_c]^{(e)}\{q_c(t)\}^{(e)} = \{0\} \tag{3}$$

Where $\{q_c(t)\}^{(e)}$ is the nodal dofs of the crack element, the subscript *c* represents the *crack* and $[K_c]^{(e)}$ is the stiffness matrix of the cracked element and it is given as

$$[K_c]^{(e)}\,[T][C]^{(e)^{-1}}[T]^T \qquad \text{with} \quad [T]^T = \begin{bmatrix} -1 & 0 & 0 & -l & 1 & 0 & 0 & 0 \\ 0 & -1 & l & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{4}$$

Where *[C]* is the flexibility matrix of the cracked element alone and *[T]* is the transformation matrix. The flexibility matrix *[C]* can be written as the sum of the uncracked element flexibility matrix $[C_0]^{(e)}$ amd the crack flexibility matrix $[C_c]^{(e)}$, hence

$$[C]^{(e)} = [C_0]^{(e)} + [C_c]^{(e)} \qquad \text{with} \qquad [C_0]^{(e)} = \begin{bmatrix} l^3/3EI & 0 & 0 & l^2/2EI \\ 0 & l^3/3EI & -l^2/2EI & 0 \\ 0 & -l^2/2EI & 1/EI & 0 \\ l^2/2EI & 0 & 0 & 1/EI \end{bmatrix} \tag{5}$$

Equation of motion of the complete system can be obtained by assembling the contribution of all equations of motion for cracked and uncracked elements in the system. Then the system equation of motion becomes

$$[M]\{\ddot{q}(t)\} + [K]\{q(t)\} = \{0\} \tag{6}$$

where $[M]$ is the assembled mass matrix, $[K]$ is the assembled stiffness matrix and $\{q(t)\}$ is the assembled vector of nodal dofs.

4. Free Vibrations

For obtaining natural frequencies, reduced system of equations of motion after applying boundary conditions to equation (6)

$$[\overline{M}]\{\ddot{\bar{q}}(t)\} + [\overline{K}]\{\bar{q}(t)\} = \{0\} \tag{7}$$

where $[\overline{M}]$ and $[\overline{K}]$ are the reduced assembled mass and stiffness matrices. By assuming the solution of the form = $\{\bar{q}(t)\} = \{\overline{Q}\}e^{j\omega_n t}$, where ω_n is the natural frequency and $\{\overline{Q}\}$ is the corresponding mode shape and substitution into equation (7), the associated eigen value problem becomes

$$([\overline{Z}] - p[I])\{\overline{Q}\} = \{0\} \tag{8}$$

where $[\overline{Z}] = [\overline{K}]^{-1}[\overline{M}]$ is known as the dynamic matrix. Equation (8) contains *n* roots (p_r; $r = 1, 2, \ldots, n$) and p_r is related to the natural frequencies by $p_r = 1/\omega_r^2$. if $\{\overline{Q}_r\}$ represents the eigenvector (mode shape) corresponding to the eigenvalue p_r, then *n* eigenvectors be obtained as

$$([\overline{Z}] - p_r[I])\{\overline{Q}\} = \{0\} \qquad \text{with } r = 1, 2, \ldots\ldots, n \tag{9}$$

5. Identification algorithm

In equation (6) the assembled stiffness matrix $[K]$ is split as $[K] = [K_{wc}] + [K_c]$, where $[K_{wc}]$ and $[K_c]$ are of the same size as that of the assembled stiffness matrix. The matrix $[K_c]$ contains contribution at cracked element dofs from the cracked element only whereas the matrix $[K_{wc}]$ contains contribution of all other elements. On substituting into equation (6) the resulting equation, after rearranging, becomes

$$[K_c]\{Q\} = \{\hat{F}\} \qquad \text{with} \qquad \{\hat{F}\} = \left(\omega_n^2[M] - [K_{wc}]\{Q\}\right) \tag{10}$$

Since the matrix $[K_c]$ is of the same size as the assembled stiffness matrix and it contains non-zero terms only corresponding to the crack nodal dofs. Thus, equation (10) can be reduced to the following form

$$[K_c]^{(e)}\{Q_c\}^{(e)} = \{\hat{F}_c\} \tag{11}$$

where $\{\hat{F}_c\}$ contains elements only corresponding to cracked element nodal dofs in $\{\hat{F}\}$ as defined in equation (10). Equation (11) can be used to obtain the unknown flexibility coefficients in the matrix $[K_c]^{(e)}$. However, since the matrix $[K_c]^{(e)}$ contains the flexibility coefficient matrix $[C_c]^{(e)}$ in the form of its inverse hence the resulting identification algorithm becomes a non-linear estimator and that is having inherent problem of convergence along with the problem of obtaining the physically meaningful parameters. The non-linear estimator has been avoided in the present study with the following re-arrangement of equation (11). On post-multiplication in both sides of equation (4) by response $\{Q_c\}^{(e)}$ yields

$$[K_c]^{(e)}\ \{Q\}^{(e)} = [T][C]^{(e)^{-1}}[T]^T\{Q_c\}^{(e)} \tag{12}$$

On equating Equations (11) and (12), we get

$$[T][C]^{(e)^{-1}}[T]^T\{Q_c\}^{(e)}=\{\hat{F}_c\} \tag{13}$$

By rearranging equation (13), we get

$$[C_c]^{(e)}\{A_1\}= \{B_2\} \tag{14}$$

with $\{B_2\}=\{B_1\}-[C_0]^{(e)}\{A_1\};\qquad \{A_1\}= ([T]^T[T])^{-1}[T]^T\{\hat{F}_c\}$ and $\{B_1\} = [T]^T[Q_c])^{(e)}$ (15)

Equation (14) can be rearranged in the standard regression equation as

Equation (14) can be rearranged in the standard regression equation as

$$[S]\{C\}= \{B_2\}\quad \text{with } \{C\}=\{C_{22}\quad C_{33}\quad C_{44}\quad C_{45}\quad C_{55}\} \tag{16}$$

In equation (16), the vector $\{C\}$ contains all unknown crack flexibility coefficients. The matrix $[S]$and the vector $\{B_2\}$ contain all known information i.e. the uncracked beam model, the crack location, natural frequencies and corresponding mode shapes. Equation (16) contains four equations with five unknown crack flexibility coefficients; hence it is an underdetermined system of equations. Since crack flexibility coefficients do not change with modes, hence, with minimum up to second modes we can have eight equations to solve for five unknowns. Then the system of equation will be an over determined which could be solved using the normal least squares method. Theoretically, from the fracture mechanics approach flexibility coefficients of the compliance matrix are expressed as a function of crack depth ratio $\bar{a} = a/R$ [7]. The error function between the identified (superscript: *id*) flexibility coefficients and the theoretical (superscript: *th*) flexibility coefficients can be defined as,

$$\pi_{error} = \sum_{i=2}^{5}\left(C_{ii}^{th} - C_{ii}^{id}\right)^2 + \left(C_{45}^{th} - C_{45}^{id}\right)^2 \tag{17}$$

where C_{ij} are crack flexibility coefficients. Minimizing the error function with respect to the crack depth ratio $\bar{a}$ in conjunction with Newton-Raphson method, the equivalent crack depth ratio can be obtained.

6. Numerical examples

For the numerical example the assumed properties of the beam are presented in the Table1. The first and second natural frequencies of uncracked beam are found from the equation (8) as 20.18 Hz and 80.73 Hz, respectively. The corresponding mode shapes are shown in Figure 2 with solid lines. A crack of the crack of the depth ratio $\bar{a} = a/R = 0.7$ is introduced at the crack location ratio $\bar{x} = x/L = 0.6$. Crack flexibility coefficients are obtained from the linear fracture mechanics approach [7]. By using equation (8) the first and second natural frequencies of the cracked beam are found to be 19.97 Hz and 80.3 Hz. Mode shapes associated with these frequencies are also obtained and shown in Figure 2 with dashed lines. These natural frequencies and mode shapes, which are obtained from the numerical experiment, is used to estimate crack parameters. Through equation (16) crack flexibility coefficients are estimated. The estimated crack flexibility coefficients are given in Table 2. The equivalent crack depth is estimated as 0.698 from equation (17). For other crack sizes also, estimated parameters are presented in the Table 2 and all the estimated parameters are very close to assumed crack parameters.

7. Conclusions

In order to obtain the crack parameters (i.e. crack flexibility coefficients and crack depth) of a cracked beam, an identification algorithm has been described. The crack flexibility coefficients are estimated from the measured free vibration i.e. the natural frequencies and mode shapes of the cracked beam. The equivalent crack depth is obtained by using the crack flexibility coefficients. Since the FEM is applied in the proposed algorithm, it has flexibility over the types of constraints (i.e. support conditions) and kinds of loads applied to the system. Hence it is quite general in nature and simple to apply. Numerical examples illustrate and validate the method. As it is difficult to measure mode shapes accurately in actual systems, the practicability of the method is needed further investigations.

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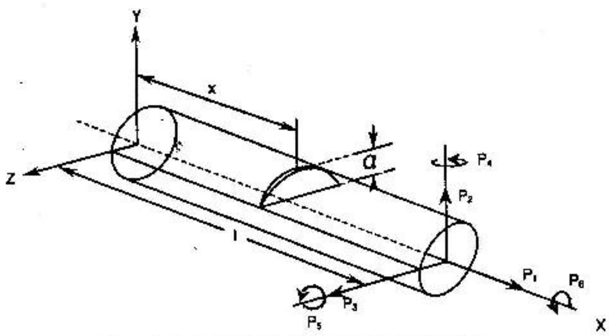

Figure 1. A cracked beam element in a general loading.

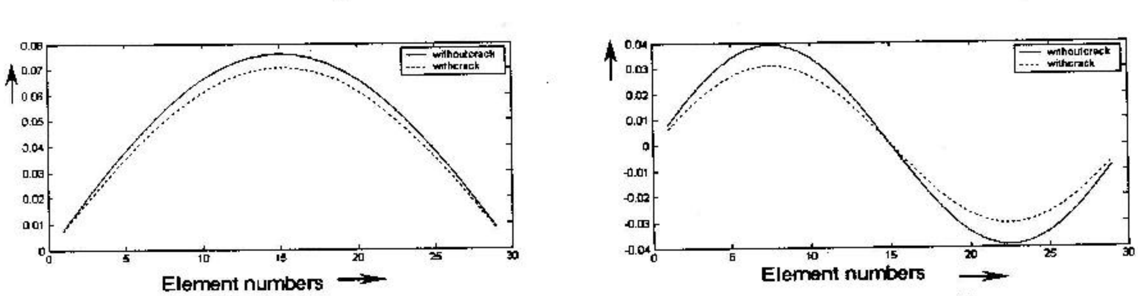

Figure 2. (a) First mode shape and (b) second mode shape of the uncracked and cracked beam

Table 1. parameters used in numerical experiments

Parameters	Values
Diameter of the circular beam, D	0.01m
Beam length, L	1.0 m
Young's modulus of the beam meterial, E	2.06×10^{11}N/m^2
Density of the beam material, ρ	7800kg/m^3
Number of beam elements, N	30
Crack location ratio, x_c/L	0.6

Table 2. The assumed and identified crack parmeters and their details

S. No	Assumed crack depth ratio	First natural frequency in Hz	Second natural frequency in Hz	Crack Parameters										
				Crack flexibility coefficients (dimensionless)										Equivalent crack depth ratio in m/m
				C_{22}		C_{33}		C_{44}		$C_{45}(=C_{54})$		C_{55}		
				Assumed*	Identified	Assumed	Identified	Assumed	Identified	Assumed	Identified	Assumed	Identified	
1	0.1	20.179	80.439	0.00241	0.00241	0.00359	0.00359	0.00134	0.00134	0.01257	0.01257	0.08725	0.08725	0.100
2	0.2	20.172	80.438	0.01351	0.01351	0.02017	0.02017	0.01494	0.01494	0.09517	0.09517	0.45432	0.45432	0.201
3	0.3	20.156	80.437	0.03696	0.03696	0.05524	0.05524	0.06170	0.06170	0.30788	0.30788	1.16935	1.16935	0.300
4	0.4	20.131	80.432	0.07547	0.07547	0.11285	0.11285	0.17146	0.17146	0.71003	0.71003	2.27728	2.27728	0.396
5	0.5	20.094	80.418	0.13139	0.13139	0.19651	0.19651	0.38746	0.38746	1.37245	1.37245	3.83756	3.83756	0.497
6	0.6	20.042	80.381	0.20702	0.20702	0.30966	0.30966	0.77652	0.77652	2.39336	2.39336	5.94451	5.94451	0.598
7	0.7	19.969	80.299	0.30482	0.30482	0.45604	0.45604	1.44912	1.44912	3.92196	3.92196	8.74793	8.74793	0.698
8	0.8	19.869	80.162	0.42765	0.42765	0.64011	0.64011	2.60080	2.60080	6.19836	6.19836	12.4831	12.4831	0.798
9	0.9	19.728	79.956	0.57910	0.57910	0.86779	0.86779	4.60569	4.60569	9.62947	9.62947	17.5223	17.5223	0.898

* The assumed crack flexibility coefficients are obtained from the linear fracture mechanics approach.