The Use of Higher Order Base Functions in Element Free Galerkin Method

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Abstract :

Element free Galerkin (EFG) method is one kind of meshfree methods that for solving partial differential equations with moving least square technique. EFG method requires only nodal data and no element connectivity is needed. In a previous implementation of the EFG method, normally, quadratic polynomial is employed to solve Euler-Bernoulli beam problem. In this paper, studies have been carried out with higher order of polynomial in the base functions of EFG method. Numerical examples show that the present implementation increases the speed of convergence and the accuracy of EFG. The comparison with published results shows improvement in computing natural frequencies and mode shapes of cantilever beams.

Keyword Element Free Galerkin (EFG) method, Higher Order Base Functions, Euler-Bernoulli Beam, Natural Frequency.

1. Introduction

Element Free Galerkin (EFG) method was proposed 20 years ago by Belytschko, T. [1, 2], however, it is only recently it captured the interest of researchers. Certain classes of problems present difficulties when solved using conventional FEM (Finite Element Method). The difficulties arise due to the inherent structure of the finite element: the rigid connectivity defined by elements. In the modeling of large deformation processes, considerable loss in accuracy arises when the elements in the mesh become extremely skewed or compressed. The traditional technique for handling these complications is to remesh the domain of the problem at every step during the evolution of the simulation. This prevents the severe distortion of elements. To ameliorate these difficulties, a new class of methods have recently been developed which do not require a mesh to discretize the problem. One of this kind of methods is Element Free Galerkin (EFG) method. The EFG method requires only a set of nodes scattered within the problem domain as well as sets of nodes scattered on the boundaries of the domain. The shape functions are constructed using moving least square (MLS) approximation which has been developed for curve and surface fitting of random data [3].

2. EFG Method

EFG method employs moving least-square (MLS) approximants to approximate the function u(x) with $u^h(x)$. These approximants are constructed from three components: weight function of compact support associated with each node, a basis usually consisting of a polynomial, and a set of coefficients that depend on the positions.

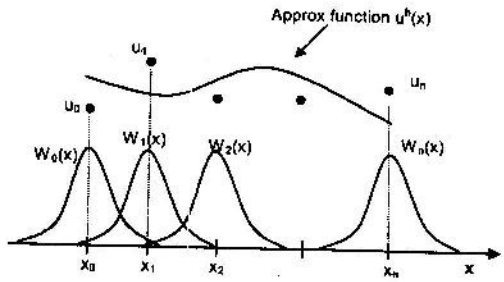

Figure 1: Moving Least square method

The MLS approximation $u^h(x)$ is given by

$$u^h(\mathbf{x}) = \sum_{j=1}^{m} p_j(\mathbf{x}) a_j(\mathbf{x}) \equiv p^T(\mathbf{x})\mathbf{a}(\mathbf{x}) \tag{1}$$

where $p(\mathbf{x})$ is a complete polynomial basis of arbitrary order and $a(\mathbf{x})$ are coefficients which are functions of the space coordinates x. Further, m is the number of terms of polynomial basis.

Examples of bases in one dimension are:

$$p^T(x) = p^T(x) = \{1, x, x^2, ..., x^n\} \tag{2}$$

$$p^T(x) = \{1, x\} \quad (m=2, \text{ linear}) \tag{3}$$

$$p^T(x) = \{1, x, x^2\} \quad (m=3, \text{ quadratic}) \tag{4}$$

For EFG, the weight function $w(x-x_i)$ is generally chosen as a monotonically decreasing function as $\|x - x_i\|$ increases. Defining $d_i = \|x - x_i\|$, and $r=d_i/d_{mi}$, where d_{mi} is the radius of the support domain, the weight function can be written more compactly as a function of the normalized distance r.

Some examples of membership functions are shown by using circle support domain.

Quadratic polynomial: $$w(r) = \begin{cases} 1 - 6r^2 + 8r^3 - 3r^4 & \text{for } r \le 1 \\ 0 & \text{for } r > 1 \end{cases} \tag{5}$$

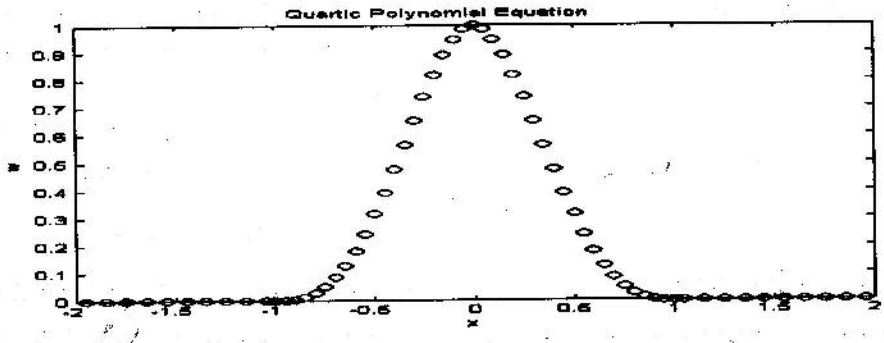

Figure 2 Quartic Polynomial Weight Function

The coefficients $a(\mathbf{x})$ in Equation (1) are obtained at any point $\mathbf{x}$ by minimizing the following weighted, discrete error norm:

$$J = \sum_{i}^{n} w(\mathbf{x} - \mathbf{x}_i)\left[u^h(\mathbf{x}, \mathbf{x}_i) - u(\mathbf{x}_i)\right]^2 = \sum_{i}^{n} w(\mathbf{x} - \mathbf{x}_i)\left[p^T(\mathbf{x}_i)\mathbf{a}(\mathbf{x}) - u(\mathbf{x}_i)\right]^2 \tag{6}$$

where $u(x_i)$ is the nodal value associated with node i at $x=x_i$ (for the beam, $u(x_i)$ is the nodal deflection) , n is the number of nodes in the neighborhood of point x (usually x is Gauss points) for which the weight function $w(x-x_i)=0$ as shown in Figure 3.

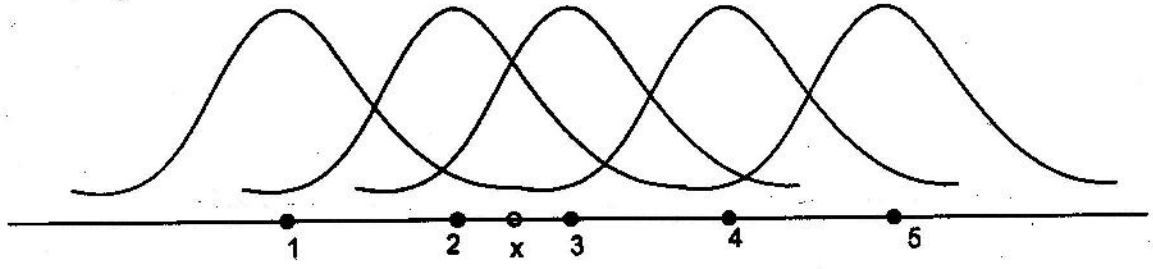

Figure 3 Overlapping domains of support and local node numbering at point x.

The EFG shape functions are constructed by minimizing $J(\mathbf{x})$. Returning to the weighted discrete error norm in Equation (6), the stationary value of $J(\mathbf{x})$ with respect to $\mathbf{a}(\mathbf{x})$ leads to

$$\frac{\partial J(\mathbf{x})}{\partial \mathbf{a}(\mathbf{x})} = 0 \tag{7}$$

which results in the following linear equation system:

$$A(\mathbf{x})\mathbf{a}(\mathbf{x}) = C(\mathbf{x})U \tag{8}$$

where A is called the weighted moment matrix given by

$$A(\mathbf{x}) = \sum_{i=1}^{n} w(\mathbf{x}-\mathbf{x}_i)p(\mathbf{x}_i)p^T(\mathbf{x}_i) \tag{9}$$

In the Equation (8), matrix C has the form of

$$C(\mathbf{x}) = [w(\mathbf{x}-\mathbf{x}_1)p(\mathbf{x}_1), w(\mathbf{x}-\mathbf{x}_2)p(\mathbf{x}_2),..., w(\mathbf{x}-\mathbf{x}_n)p(\mathbf{x}_n)] \tag{10}$$

Noting that U is the vector that collects the nodal parameters of the field variable for all nodes whose support domain includes **x**

$$U = [u_1, u_2,..., u_n]^T \tag{11}$$

and solving the Equation (8) for **a(x)**, results in

$$\mathbf{a}(\mathbf{x}) = A^{-1}(\mathbf{x})C(\mathbf{x})U \tag{12}$$

By substituting (13) into (1), the MLS approximants are

$$u^h(\mathbf{x}) = \sum_{i=1}^{n}\sum_{j=1}^{m} p_j(\mathbf{x})(A^{-1}(\mathbf{x})C(\mathbf{x}))_{ji}\, u_i \tag{13}$$

Equation (13) can be rewritten:

$$u^h(\mathbf{x}) = \sum_{i=1}^{n} \varphi_i(\mathbf{x})u_i \tag{14}$$

where the EFG shape functions φ are defined as

$$\varphi_i(\mathbf{x}) = \sum_{j=1}^{m} p_j(\mathbf{x})(A^{-1}(\mathbf{x})C(\mathbf{x}))_{ji} = p^T(\mathbf{x})A^{-1}(\mathbf{x})C_i \tag{15}$$

Equation (15) can be rewritten as:

$$u^h(\mathbf{x}) = \mathbf{\Phi}(\mathbf{x})U \tag{16}$$

where $\mathbf{\Phi}(\mathbf{x})$ is the matrix of MLS shape functions corresponding to nodes whose support of domain covers point x.

$$\mathbf{\Phi}(\mathbf{x}) = [\varphi_1(\mathbf{x}), \varphi_2(\mathbf{x}), \cdots, \varphi_n(\mathbf{x})] \tag{17}$$

Since shape functions are at hands, the next step is calculating nodal stiffness and nodal mass matrix. Once the global discretized system equation is established, EFG method follows a procedure similar to FEM. The formula of nodal stiffness matrix and nodal mass matrix are as follows:

$$K_{ij} = \int B_i^T EI_z B_j dx \tag{18}$$

B_i is the second derivative of the shape function

$$B_i = \frac{\partial^2 \varphi_i(x)}{\partial x^2} \tag{19}$$

$$I_z = \int_A y^2 dA \tag{20}$$

The dimension of nodal stiffness matrix K_{ij} is 1x1.

$$M_{ij} = \int_\Omega \varphi_i^T \rho \varphi_j d\Omega = \int_0^l \varphi_i^T \rho \varphi_j A dx = \int_0^l \rho A \varphi_i^T \varphi_j dx \tag{21}$$

M_{ij} is called nodal mass matrix, the dimension being 1×1.

After nodal stiffness matrix and nodal mass matrix are obtained, we should assemble them into global stiffness matrix K and global mass matrix M. Apply boundary conditions by using the constraint matrix H and Singular Value Decomposition [4] method to a cantilever beam. Then, we solve the eigenvalue equations as follows:

$$(K - \omega^2 M)U = 0 \tag{22}$$

3. Increase the order of polynomial in base function

Normally, Quadratic polynomial has been found in many EFG papers to solve beam problem. The reason is that the governing weak form for beam contains second-order derivatives, a quadratic polynomial must be employed for the purpose of consistency. In this study, higher order polynomial base functions are tried to solve beam problem by EFG. The higher order polynomial base functions are shown as follows:

$$p^T(x) = \{1, x, x^2, x^3\} (m=4) \tag{23}$$

$$p^T(x) = \{1, x, x^2, x^3, x^4\} (m=5) \tag{24}$$

4. Numerical Results

The natural frequencies of the cantilever beam are calculated using Element Free Galerkin (EFG) method. The results are compared with the exact solution. Comparing mode shapes with those obtained by analytical methods, good agreements have been noted. The parameters of the beam are Young's modulus E = 2.1e+11 N/m², Mass Density ρ = 7800 kg/m³, Thickness b = 0.01 m, Height h = 0.1 m, Length L = 1 m. The following figures show the results of 11 nodes cantilever beam. Different polynomial base functions are used.

m=3, $p^T(x) = \{1, x, x^2\}$

m=4, $p^T(x) = \{1, x, x^2, x^3\}$

m=5, $p^T(x) = \{1, x, x^2, x^3, x^4\}$

First five natural frequencies of EFG are compared with the analytical solutions. Relative errors are computed by:

$$\text{error} = \frac{\omega_{EFG} - \omega_{EXACT}}{\omega_{EXACT}}$$

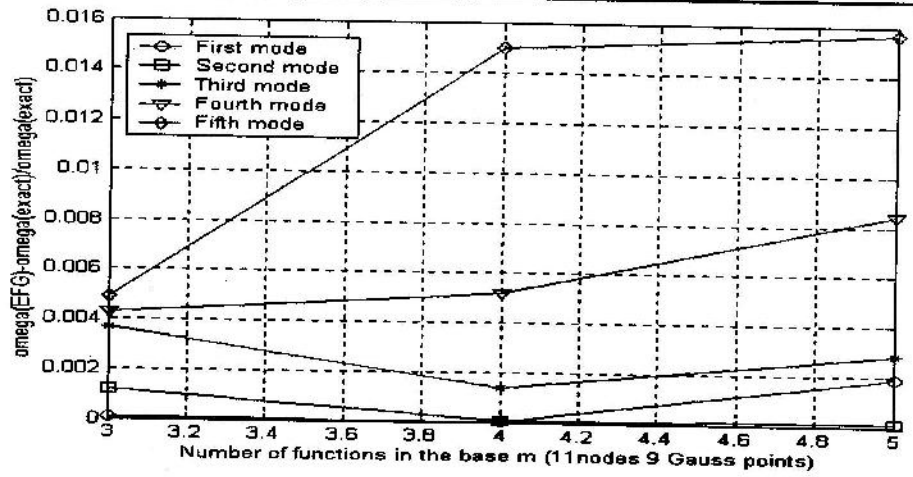

Figure 4 Frequency relative error for 11- node beam subjected to bending vibration with different polynomial base functions (9 Gauss Points in each cell, 10 cells are used)

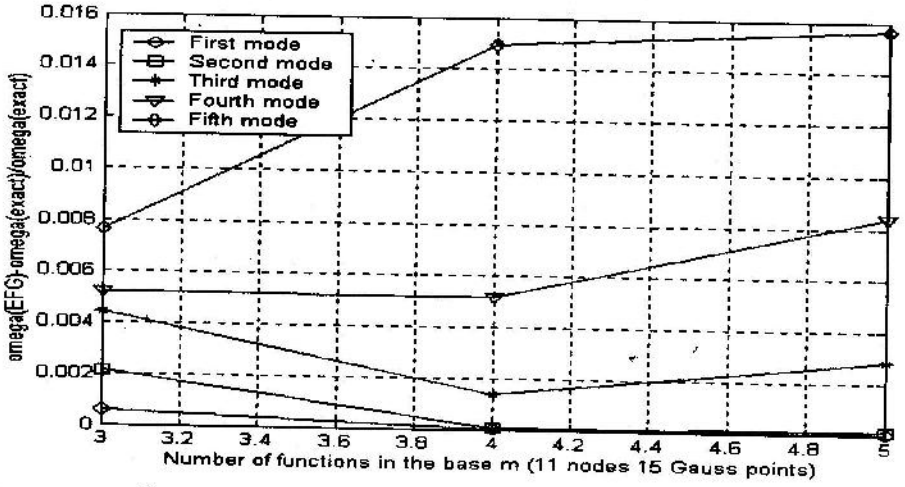

Figure 5 Frequency relative error for 11- node beam subjected to bending vibration with different polynomial base functions (15 Gauss Points in each cell, 10 cells are used)

From Figure 4 and Figure 5, when 11 nodes cantilever beam is used, higher order polynomial base functions (m=4, m=5) can not give better results even though Gauss points in each cell are increased. Because Gauss points have nothing to do with the construction of shape functions, and Gauss points just contribute to the accuracy of integration. The number of nodes distributed along the beam directly affects building shape function. So we increase the number of nodes when we try to use higher order of polynomials.

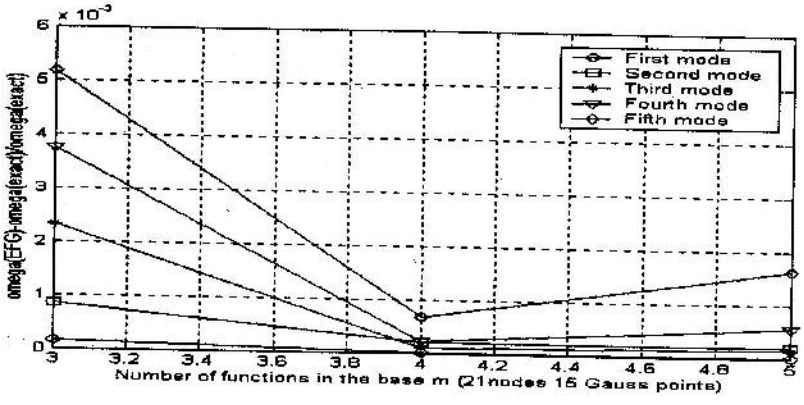

Figure 6 Frequency relative error for 21- node beam subjected to bending vibration with different polynomial base functions (15 Gauss Points in each cell, 20 cells are used)

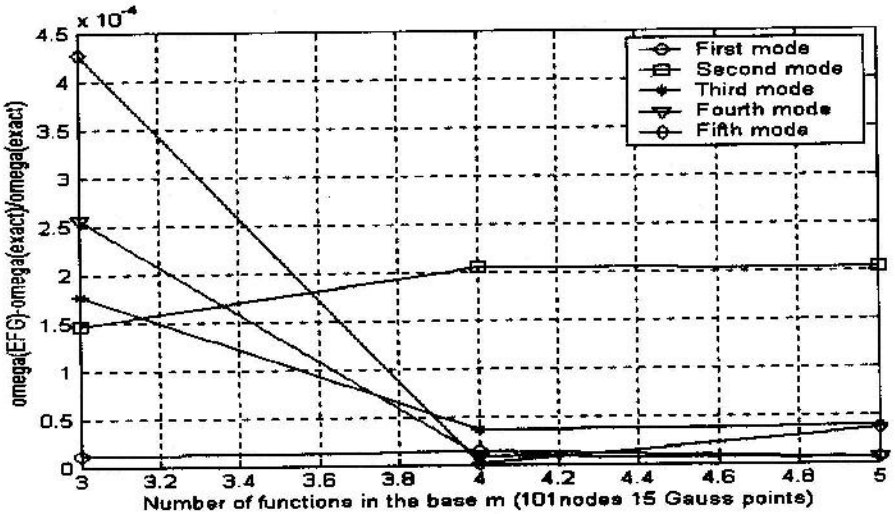

Figure 7 Frequency relative error for 101- node beam subjected to bending vibration with different polynomial base functions (15 Gauss Points in each cell, 100 cells are used)

From Figure 6 and Figure 7, the advantage of using higher order polynomial base functions is obvious. In Figure 6, errors are reduced greatly from m=3 to m=4. m=4 is the best choice when 21- node model is used. In Figure 7, when the number of nodes is increased to 101, the error of m=3 and m=4 are almost same, the maximum error is smaller ($\approx 2x10^{-4}$) compared with the maximum error of Figure 6 ($\approx 7x10^{-4}$).

From numerical experiment, when higher order polynomial base functions are chosen to build shape functions, the radius of each support domain is larger than that of lower order polynomial base functions, because the higher order polynomial base functions need much more nodes to make an approximation.

In order to achieve higher accuracy by using higher order polynomial base functions, we need increased the radius of each support domain and the number of Gauss point in each cell at the same time. Too small number of nodes cannot show the advantage of higher order polynomial base functions (For example,11-node cantilever beam).

5. Conclusions

This paper effectively improved the results in EFG method by applying higher order basis functions. Increase in the order of polynomial in the base function of EFG, increases the speed of convergence and the accuracy of EFG method. When the order of polynomial is increased, the radius of support of each node and the Gauss points in each cell should be increased at the same time in order to achieve the higher accuracy.

Reference

[1] T. Belytschko, Y. Krongauz, D. Organ, M. Fleming and P. Krysl (1996). Mesh- less Methods: An Overview and Recent Developments. Computer Methods in Applied Mechanics and Engineering 139, 3-47.

[2] T. Belytschko, Y. Y. Lu, and L. Gu (1994), "Element Free Galerkin Methods", International Journal for Numerical Methods in Engineering 37, 229-256.

[3] P. Lancaster, and K. Salkauskas (1981), "Surfaces Generated by Moving Least Squares Methods", Mathematics of Computation 37, 141-158.

[4] A. E. Ouatouati and D. A. Johnson (1999), "New Approach for Numerical Modal Analysis Using the Element Free Method", International Journal for Numerical Methods in Engineering 46,1-27.